


Partial Correctness of a Factorial Algorithm

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Summary. In this paper we present a formalization in the Mizar system [3],[1] of the partial correctness of the algorithm:

```
i := val.1
j := val.2
n := val.3
s := val.4
while (i <> n)
  i := i + j
  s := s * i
return s
```

computing the factorial of given natural number n , where variables i , n , s are located as values of a V -valued Function, loc , as: $loc/.1 = i$, $loc/.3 = n$ and $loc/.4 = s$, and the constant 1 is located in the location $loc/.2 = j$ (set V represents simple names of considered nominative data [16]).

This work continues a formal verification of algorithms written in terms of simple-named complex-valued nominative data [6],[8],[14],[10],[11],[12]. The validity of the algorithm is presented in terms of semantic Floyd-Hoare triples over such data [9]. Proofs of the correctness are based on an inference system for an extended Floyd-Hoare logic [2],[4] with partial pre- and post-conditions [13],[15],[7],[5].

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Let D be a set and f_1, f_2, f_3 be binominative functions of D . The functor $\text{PP-composition}(f_1, f_2, f_3)$ yielding a binominative function of D is defined by the term

(Def. 1) $f_1 \bullet f_2 \bullet f_3$.

Let f_1, f_2, f_3, f_4 be binominative functions of D . The functor PP-composition (f_1, f_2, f_3, f_4) yielding a binominative function of D is defined by the term

(Def. 2) $\text{PP-composition}(f_1, f_2, f_3) \bullet f_4$.

From now on D denotes a non empty set, f_1, f_2, f_3, f_4 denote binominative functions of D , and p, q, r, t, w denote partial predicates of D .

Now we state the proposition:

(1) UNCONDITIONAL COMPOSITION RULE FOR 3 PROGRAMS:

Suppose $\langle p, f_1, q \rangle$ is an SFHT of D and $\langle q, f_2, r \rangle$ is an SFHT of D and $\langle r, f_3, w \rangle$ is an SFHT of D and $\langle \sim q, f_2, r \rangle$ is an SFHT of D and $\langle \sim r, f_3, w \rangle$ is an SFHT of D . Then $\langle p, \text{PP-composition}(f_1, f_2, f_3), w \rangle$ is an SFHT of D .

(2) UNCONDITIONAL COMPOSITION RULE FOR 4 PROGRAMS:

Suppose $\langle p, f_1, q \rangle$ is an SFHT of D and $\langle q, f_2, r \rangle$ is an SFHT of D and $\langle r, f_3, w \rangle$ is an SFHT of D and $\langle w, f_4, t \rangle$ is an SFHT of D and $\langle \sim q, f_2, r \rangle$ is an SFHT of D and $\langle \sim r, f_3, w \rangle$ is an SFHT of D and $\langle \sim w, f_4, t \rangle$ is an SFHT of D . Then $\langle p, \text{PP-composition}(f_1, f_2, f_3, f_4), t \rangle$ is an SFHT of D .

In the sequel d, v, v_1 denote objects, V, A denote sets, z denotes an element of V , d_1 denotes a non-atomic nominative data of V and A , f denotes a binominative function over simple-named complex-valued nominative data of V and A , and T denotes a nominative data with simple names from V and complex values from A .

Now we state the proposition:

(3) If V is without nonatomic nominative data w.r.t. A and $v \in V$ and $v \neq v_1$ and $v_1 \in \text{dom } d_1$, then $(d_1 \nabla_a^v T)(v_1) = d_1(v_1)$.

Let x, y be objects. Assume x is a complex number and y is a complex number. The functors: $x + y$ and $x * y$ yielding complex numbers are defined by conditions

(Def. 3) there exist complex numbers x_1, y_1 such that $x_1 = x$ and $y_1 = y$ and $x + y = x_1 + y_1$,

(Def. 4) there exist complex numbers x_1, y_1 such that $x_1 = x$ and $y_1 = y$ and $x * y = x_1 \cdot y_1$,

respectively. Let us consider A . Assume A is complex containing. The functors: $\text{addition}(A)$ and $\text{multiplication}(A)$ yielding functions from $A \times A$ into A are defined by conditions

(Def. 5) for every objects x, y such that $x, y \in A$ holds $\text{addition}(A)(x, y) = x + y$,

(Def. 6) for every objects x, y such that $x, y \in A$ holds $\text{multiplication}(A)(x, y) = x * y$,

respectively. Let us consider V . Let x, y be elements of V . The functors: addition

(A, x, y) and $\text{multiplication}(A, x, y)$ yielding binominative functions over simple-named complex-valued nominative data of V and A are defined by terms

(Def. 7) $\text{lift-binary-op}(\text{addition}(A), x, y)$,

(Def. 8) $\text{lift-binary-op}(\text{multiplication}(A), x, y)$,

respectively.

Let us consider elements i, j of V and complex numbers x, y . Now we state the propositions:

(4) Suppose A is complex containing and $i, j \in \text{dom } d_1$ and $d_1 \in \text{dom}(\text{addition}(A, i, j))$. Then if $x = d_1(i)$ and $y = d_1(j)$, then $(\text{addition}(A, i, j))(d_1) = x + y$.

(5) Suppose A is complex containing and $i, j \in \text{dom } d_1$ and $d_1 \in \text{dom}(\text{multiplication}(A, i, j))$. Then if $x = d_1(i)$ and $y = d_1(j)$, then $(\text{multiplication}(A, i, j))(d_1) = x \cdot y$.

In the sequel loc denotes a V -valued function and val denotes a function.

Let us consider V , A , and loc . The functor $\text{factorial-loop-body}(A, loc)$ yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 9) $\text{Asg}^{(loc/1)}(\text{addition}(A, loc/1, loc/2)) \bullet (\text{Asg}^{(loc/4)}(\text{multiplication}(A, loc/4, loc/1)))$.

The functor $\text{factorial-main-loop}(A, loc)$ yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 10) $\text{WH}(\neg \text{Equality}(A, loc/1, loc/3), \text{factorial-loop-body}(A, loc))$.

Let us consider val . The functor $\text{factorial-var-init}(A, loc, val)$ yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 11) $\text{PP-composition}(\text{Asg}^{(loc/1)}(val(1) \Rightarrow_a), \text{Asg}^{(loc/2)}(val(2) \Rightarrow_a), \text{Asg}^{(loc/3)}(val(3) \Rightarrow_a), \text{Asg}^{(loc/4)}(val(4) \Rightarrow_a))$.

The functor $\text{factorial-main-part}(A, loc, val)$ yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 12) $\text{factorial-var-init}(A, loc, val) \bullet (\text{factorial-main-loop}(A, loc))$.

Let us consider z . The functor $\text{factorial-program}(A, loc, val, z)$ yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 13) $\text{factorial-main-part}(A, loc, val) \bullet (\text{Asg}^z((loc/4) \Rightarrow_a))$.

In the sequel n_0 denotes a natural number.

Let us consider V , A , val , n_0 , and d . We say that n_0 and d constitute a valid input for the factorial w.r.t. V , A and val if and only if

(Def. 14) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and $\{val(1), val(2), val(3), val(4)\} \subseteq \text{dom } d_1$ and $d_1(val(1)) = 0$ and $d_1(val(2)) = 1$ and $d_1(val(3)) = n_0$ and $d_1(val(4)) = 1$.

The functor $\text{valid-factorial-input}(V, A, val, n_0)$ yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 15) $\text{dom } it = \text{ND}_{\text{SC}}(V, A)$ and for every object d such that $d \in \text{dom } it$ holds if n_0 and d constitute a valid input for the factorial w.r.t. V , A and val , then $it(d) = \text{true}$ and if n_0 and d do not constitute a valid input for the factorial w.r.t. V , A and val , then $it(d) = \text{false}$.

Note that $\text{valid-factorial-input}(V, A, val, n_0)$ is total.

Let us consider z and d . We say that n_0 and d constitute a valid output for the factorial w.r.t. A and z if and only if

(Def. 16) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and $z \in \text{dom } d_1$ and $d_1(z) = n_0!$.

The functor $\text{valid-factorial-output}(A, z, n_0)$ yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 17) $\text{dom } it = \{d, \text{ where } d \text{ is a nominative data with simple names from } V \text{ and complex values from } A : d \in \text{dom}(z \Rightarrow_a)\}$ and for every object d such that $d \in \text{dom } it$ holds if n_0 and d constitute a valid output for the factorial w.r.t. A and z , then $it(d) = \text{true}$ and if n_0 and d do not constitute a valid output for the factorial w.r.t. A and z , then $it(d) = \text{false}$.

Let us consider loc and d . We say that n_0 and d constitute a valid invariant for the factorial w.r.t. A and loc if and only if

(Def. 18) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and $\{loc_{/1}, loc_{/2}, loc_{/3}, loc_{/4}\} \subseteq \text{dom } d_1$ and $d_1(loc_{/2}) = 1$ and $d_1(loc_{/3}) = n_0$ and there exist natural numbers I, S such that $I = d_1(loc_{/1})$ and $S = d_1(loc_{/4})$ and $S = I!$.

The functor $\text{factorial-inv}(A, loc, n_0)$ yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 19) $\text{dom } it = \text{ND}_{\text{SC}}(V, A)$ and for every object d such that $d \in \text{dom } it$ holds if n_0 and d constitute a valid invariant for the factorial w.r.t. A and loc , then $it(d) = \text{true}$ and if n_0 and d do not constitute a valid invariant for the factorial w.r.t. A and loc , then $it(d) = \text{false}$.

One can check that $\text{factorial-inv}(A, loc, n_0)$ is total.

Let us consider val . We say that loc and val are compatible w.r.t. 4 locations if and only if

(Def. 20) $val(4) \neq loc_{/3}$ and $val(4) \neq loc_{/2}$ and $val(4) \neq loc_{/1}$ and $val(3) \neq loc_{/2}$ and $val(3) \neq loc_{/1}$ and $val(2) \neq loc_{/1}$.

Now we state the propositions:

- (6) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $loc_{/1}, loc_{/2}, loc_{/3}, loc_{/4}$ are mutually different and loc and val are compatible w.r.t. 4 locations. Then $\langle \text{valid-factorial-input}(V, A, val, n_0), \text{factorial-var-init}(A, loc, val), \text{factorial-inv}(A, loc, n_0) \rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$.
- PROOF: Set $i = loc_{/1}$. Set $j = loc_{/2}$. Set $n = loc_{/3}$. Set $s = loc_{/4}$. Set $i_1 = val(1)$. Set $j_1 = val(2)$. Set $n_1 = val(3)$. Set $s_1 = val(4)$. Set $I = \text{valid-factorial-input}(V, A, val, n_0)$. Set $i_2 = \text{factorial-inv}(A, loc, n_0)$. Set $D_1 = i_1 \Rightarrow_a$. Set $D_2 = j_1 \Rightarrow_a$. Set $D_3 = n_1 \Rightarrow_a$. Set $D_4 = s_1 \Rightarrow_a$. Set $S_1 = \text{Sp}(i_2, D_4, s)$. Set $R_1 = \text{Sp}(S_1, D_3, n)$. Set $Q_1 = \text{Sp}(R_1, D_2, j)$. Set $P_1 = \text{Sp}(Q_1, D_1, i)$. $I \models P_1$. \square
- (7) Suppose V is not empty and A is complex containing and V is without nonatomic nominative data w.r.t. A and $loc_{/1}, loc_{/2}, loc_{/3}, loc_{/4}$ are mutually different. Then $\langle \text{factorial-inv}(A, loc, n_0), \text{factorial-loop-body}(A, loc), \text{factorial-inv}(A, loc, n_0) \rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (3), (4), and (5).
- (8) $\langle \sim \text{factorial-inv}(A, loc, n_0), \text{factorial-loop-body}(A, loc), \text{factorial-inv}(A, loc, n_0) \rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$.
- (9) Suppose V is not empty and A is complex containing and V is without nonatomic nominative data w.r.t. A and $loc_{/1}, loc_{/2}, loc_{/3}, loc_{/4}$ are mutually different. Then $\langle \text{factorial-inv}(A, loc, n_0), \text{factorial-main-loop}(A, loc), \text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{factorial-inv}(A, loc, n_0) \rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (7) and (8).
- (10) Suppose V is not empty and A is complex containing and V is without nonatomic nominative data w.r.t. A and $loc_{/1}, loc_{/2}, loc_{/3}, loc_{/4}$ are mutually different and loc and val are compatible w.r.t. 4 locations. Then $\langle \text{valid-factorial-input}(V, A, val, n_0), \text{factorial-main-part}(A, loc, val), \text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{factorial-inv}(A, loc, n_0) \rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (6) and (9).
- (11) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and for every T , T is a value on $loc_{/1}$ and T is a value on $loc_{/3}$. Then $\text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{factorial-inv}(A, loc, n_0) \models \text{Sp}(\text{valid-factorial-output}(A, z, n_0), (loc_{/4}) \Rightarrow_a, z)$.
- PROOF: Set $i = loc_{/1}$. Set $j = loc_{/2}$. Set $n = loc_{/3}$. Set $s = loc_{/4}$. Set $D_4 = s \Rightarrow_a$. Consider d_1 being a non-atomic nominative data of V and A such that $d = d_1$ and $\{i, j, n, s\} \subseteq \text{dom } d_1$ and $d_1(j) = 1$ and $d_1(n) = n_0$ and there exist natural numbers I, S such that $I = d_1(i)$ and $S = d_1(s)$ and $S = I!$. Reconsider $d_2 = d$ as a nominative data with simple names from

- V and complex values from A . Set $L = d_2 \nabla_a^z D_4(d_2)$. n_0 and L constitute a valid output for the factorial w.r.t. A and z . \square
- (12) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and for every T , T is a value on $loc_{/1}$ and T is a value on $loc_{/3}$. Then $\langle \text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{factorial-inv}(A, loc, n_0), \text{Asg}^z((loc_{/4}) \Rightarrow_a), \text{valid-factorial-output}(A, z, n_0) \rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (11).
- (13) Suppose for every T , T is a value on $loc_{/1}$ and T is a value on $loc_{/3}$. Then $\langle \sim (\text{Equality}(A, loc_{/1}, loc_{/3}) \wedge \text{factorial-inv}(A, loc, n_0)), \text{Asg}^z((loc_{/4}) \Rightarrow_a), \text{valid-factorial-output}(A, z, n_0) \rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$.
- (14) PARTIAL CORRECTNESS OF A FACTORIAL ALGORITHM:
Suppose V is not empty and A is complex containing and V is without nonatomic nominative data w.r.t. A and $loc_{/1}, loc_{/2}, loc_{/3}, loc_{/4}$ are mutually different and loc and val are compatible w.r.t. 4 locations and for every T , T is a value on $loc_{/1}$ and T is a value on $loc_{/3}$. Then $\langle \text{valid-factorial-input}(V, A, val, n_0), \text{factorial-program}(A, loc, val, z), \text{valid-factorial-output}(A, z, n_0) \rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (10), (12), and (13).

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